CERTAIN PROPERTIES OF CLASSICAL POLYNOMIALS AND THEIR APPLICATION IN CONTACT PROBLEMS

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Certain properties of Jacobi's polynomials (and, in particular, of Hegenbauer, Legendre's and Tchebysheff's polynomials) were established, and their application in contact problems was given in [1]. This paper supplements the previous results, namely: the analogous properties are established here for the polynomials of Tchebysheff-Laguerre and Tchebysheff-Hermite, and their application in constructing an approximate solution of the three-dimensional contact problem of a semi-infinite plate and the elastic half-space is given.

1. Let us note that if a linear operator L is given, such that the corresponding integral equation

$$L\varphi_m = \rho(x)\, x^m \qquad (m = 0, 1, 2, \ldots) \tag{1.1}$$

has the unique integrable solution of the form

$$\varphi_m(x) = \rho(x) \sum_{k=0}^{m} b_k^{(m)} x^k, \qquad b_m^{(m)} \neq b_k^{(k)} \neq 0 \qquad (m, k = 0, 1, 2, \ldots) \tag{1.2}$$

then a system of polynomials

$$p_m(x) = \sum_{j=0}^{m} c_j^{(m)} x^j, \qquad c_m^{(m)} = 1 \qquad (m = 0, 1, 2 \ldots) \tag{1.3}$$

which are quite simply related to the eigenfunctions of the operator L can be constructed, namely

$$L\,[p_m(x)\, \rho(x)] = \mu_m \rho(x)\, p_m(x) \qquad (m = 0, 1, 2 \ldots) \tag{1.4}$$

and, moreover

$$\mu_m = [b_m^{(m)}]^{-1} \tag{1.5}$$

The validity of the above statement may be verified in the same way as in [1]. Further, we will make use of the result of [2] in which the solution of the integral equation

$$\frac{2^{\mu}}{\sqrt{\pi}\Gamma(1/_2-\mu)}\int_0^{\infty}\frac{K_{\mu}(|t-\tau|)}{|t-\tau|^{\mu}}\varphi(\tau)\,d\tau=f(t)\qquad\left(t\geqslant 0,\ |\operatorname{Re}\mu|<\frac{1}{2}\right)\tag{1.6}$$

is constructed. Here, $K_{\mu}(z)$ is the Macdonald's function.

In particular, it follows from the above paper, that for $f(t)=e^{-\sigma t}$ the solution $\varphi_{\sigma}(t)$ of Equation (1.6) has the form

$$\varphi_{\sigma}(t)=\frac{\sin\omega\pi}{\pi}\Gamma(\omega)(1+\sigma)^{\omega}\left[\frac{e^{-t}}{t^{\omega}}+(1-\sigma)\int_0^t\frac{e^{-s}}{s^{\omega}}e^{\sigma(s-t)}\,ds\right]\qquad\left(\omega=\frac{1}{2}-\mu\right)\tag{1.7}$$

and, therefore, the solution of the following equation

$$\frac{2^{\mu}}{\sqrt{\pi}\,\Gamma(1/_2-\mu)}\int_0^{\infty}\frac{K_{\mu}(|t-\tau|)}{|t-\tau|^{\mu}}\frac{\varphi_m(\tau)}{\tau^{1/_2-\mu}}\,d\tau=e^{-t}t^m\quad(t\geqslant 0)\tag{1.8}$$

is defined by Formula

$$\varphi_m(t)=(-1)^m t^{1/_2-\mu}\left[\frac{d^m}{d\sigma^m}\varphi_{\sigma}(t)\right]_{\sigma=1}$$

Hence, after differentiating m times we find

$$\varphi_m(t)=e^{-t}\sum_{k=0}^{m}b_k^{(m)}t^k,\qquad b_k^{(m)}=\frac{2^{k-m-\mu+1}\Gamma(\mu+m-k-1/_2)\,m!}{\sqrt{2}\,\Gamma(\mu-1/_2)(m-k)!\,\Gamma(\mu+k+1/_2)}\tag{1.9}$$

Therefore, the eigenvalues μ_s of the integral operator which is contained in (1.8), in accordance with (1.5) and (1.9), will have the form

$$\mu_m=2^{\mu-1/_2}(m!)^{-1}\Gamma(1/_2+\mu+m)\tag{1.10}$$

Following the same considerations as in [1], and bearing in mind the orthogonality and normalization conditions [3] for the Tchebysheff-Laguerre polynomials $L_m^{\alpha}(x)$, one can obtain, on one hand the integral property of the above polynomials

$$\frac{1}{\sqrt{\pi}\,\Gamma(1/_2-\mu)}\int_0^{\infty}\frac{K_{\mu}(|t-\tau|)e^{-\tau}}{|t-\tau|^{\mu}\tau^{1/_2-\mu}}L_m^{\mu-1/_2}(2\tau)\,d\tau=\frac{\Gamma(1/_2+\mu+m)}{\sqrt{2}m!}e^{-t}L_m^{\mu-1/_2}(2t)$$
$$(t\geqslant 0,\ m=0,1,2\ldots)\tag{1.11}$$

and on the other hand, the expansion which converges in the mean

$$\frac{K_{\mu}(|x-y|)}{|x-y|^{\mu}}=\frac{\sqrt{\pi}\Gamma(1/_2-\mu)}{2^{-\mu}e^{x+y}}\sum_{m=0}^{\infty}L_m^{\mu-1/_2}(2x)L_m^{\mu-1/_2}(2y)\tag{1.12}$$

For $\mu=0$ the relations (1.11) and (1.12) become

$$\frac{1}{\pi}\int_0^{\infty}K_0(|t-\tau|)e^{-\tau}L_m^{-1/_2}(2\tau)\frac{d\tau}{\sqrt{\tau}}=\frac{\pi}{(2)^{1/_2}}\frac{(2m-1)!!}{2m!!}e^{-t}L_m^{-1/_2}(2t)\tag{1.13}$$

$$\frac{1}{\pi}K_0(|x-y|)=e^{-x-y}\sum_{m=0}^{\infty}L_m^{-1/_2}(2x)L_m^{-1/_2}(2y)\tag{1.14}$$

In view of the well known connection [3] between the Hermite's and Laguerre's polynomials

$$H_{2m}(\sqrt{t}) = (-1)^m 2^{2m} m!\, L_m^{-1/2}(t) \tag{1.15}$$

the latter may be replaced by the former in the relations (1.14) and (1.15), if that proves more expedient.

Moreover, since $K_0(ix) = -\frac{1}{2}\pi i H_0^{(2)}(x)$, it is easy to obtain a formal transformation of the relation (1.13) into an analogous one for the second Hankel function $H_0^{(2)}(x)$. First a substitution $t = \mu x, \tau = \mu\xi$ $(\mu > 0)$, is made in (1.13), and then the obtained result is continued analytically into the first quadrant of the complex domain of μ $(0 \leqslant \arg\mu < \frac{1}{2}\pi)$. Thus we obtain $(\mu = i\lambda)$

$$\int_0^\infty \frac{H_0^{(2)}(\lambda|x-\xi|)}{\sqrt{\xi}e^{i\lambda\xi}} L_m^{-1/2}(2i\lambda\xi)\, d\xi = \frac{2i\sqrt{\pi}}{\sqrt{2i\lambda}} \frac{(2m-1)!!}{2m!!} \frac{L_m^{-1/2}(2i\lambda x)}{e^{i\lambda x}} \tag{1.16}$$

Let us note that the integral operator contained in (1.13) gives rise to the integral equation of the first kind, to which the problem of pressing a semi-infinite punch into the elastic half-space [4] is reduced, and the integral operator contained in (1.16) plays an analogous role in the Sommerfeld's problem [5].

2. Bateman's well known result ([6], p.171), proved for the integral equations of the second kind, remains valid for the integral equations of the first kind in the following formulation.

Suppose that for Equation

$$\int_a^b k(x, y)\,\varphi(y)\,dy = f(x) \qquad (a \leqslant x \leqslant b) \tag{2.1}$$

in a certain class of functions we find the resolvent $\gamma(x, y)$, i.e. the function by which the solution of Equation (2.1) can be presented in the form

$$\varphi(x) = \int_a^b \gamma(x, y)\, f(y)\, dy \tag{2.2}$$

Then for the integral equation with the kernel

$$K_n(x, y) = \frac{1}{\Delta}\begin{vmatrix} k(x, y) & f_1^*(x) & \ldots & f_n^*(x) \\ g_1^*(y) & a_{11} & \ldots & a_{1n} \\ g_n^*(y) & a_{n1} & \ldots & a_{nn} \end{vmatrix}, \qquad \Delta = \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix} \tag{2.3}$$

where $f_k^*(x)$ and $g_s^*(y)$ are arbitrary functions, and $a_{k,s}$ are arbitrary numbers, the resolvent is given in the form

$$\Gamma_n(x, y) = \frac{1}{\Delta^*}\begin{vmatrix} \gamma(x, y) & \varphi_1^*(x) & \ldots & \varphi_n^*(x) \\ \chi_1^*(y) & \tau_{11} - a_{11} & \ldots & \tau_{1n} - a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ \chi_n^*(y) & \tau_{n1} - a_{n1} & \ldots & \tau_{nn} - a_{nn} \end{vmatrix}$$

$$\Delta^* = \begin{vmatrix} \tau_{11} - a_{11} & \ldots & \tau_{1n} - a_{1n} \\ \ldots & \ldots & \ldots \\ \tau_{n1} - a_{n1} & \ldots & \tau_{nn} - a_{nn} \end{vmatrix} \tag{2.4}$$

where

$$\varphi_m^*(x) = \int_a^b \gamma(x, y)\, f_m^*(y)\, dy, \qquad \chi_k^*(x) = \int_a^b \gamma(y, x)\, g_k^*(y)\, dy \tag{2.5}$$

$$\tau_{k,m} = \int_a^b \varphi_m^*(x)\, g_k^*(x)\, dx \tag{2.6}$$

This result, known from the theory of linear operators, is easily verified by direct substitution of function

$$\varphi_n(x) = \int_a^b \Gamma_n(x, y) f(y)\, dy$$

into Equation (2.1) with the kernel (2.3).

Consider an integral equation of the first kind with a symmetric kernel $K(x, y)$, representable in the form

$$K(x, y) = k(x, y) - K_*(x, y)$$

where $k(x, y)$ and $K_*(x, y)$ are also symmetric kernels. We will assume that for the integral operator generated by the kernel $k(x, y)$, the inverse operator is known in a certain class of functions, as well as the complete system of eigenfunctions $g_n(x)$, such that (δ_{mn} is the Kronecker's symbol)

$$\int_a^b k(x, y)\, g_m(y)\, dy = \mu_m g_m(x), \qquad \int_a^b g_m(x)\, g_n(x)\, dx = \lambda_n \delta_{mn} \tag{2.7}$$

For a sufficiently general case we can obtain the expansion

$$K_*(x, y) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_{k,m}\, g_k(x)\, g_m(y) \qquad (a_{k,m} = a_{m,k})$$

Furthermore, we introduce the designation

$$K_n(x, y) = k(x, y) - \sum_{k=0}^{n} \sum_{m=0}^{n} a_{k,m}\, g_k(x)\, g_m(y) \tag{2.8}$$

The resolvent of the integral equation of the first kind with the kernel (2.8) will be designated by $\Gamma_n(x, y)$. Assuming $\Gamma_n(x, y)$ to be known, let us construct $\Gamma_{n+1}(x, y)$, i.e. the resolvent of the integral equation of the first kind with the kernel $K_{n+1}(x, y)$. It is easy to verify that

$$K_{n+1}(x, y) = -\begin{vmatrix} K_n(x, y) & g_{n+1}(x) & S_n(x) \\ g_{n+1}(y) & 0 & 1 \\ S_n(y) & 1 & -a_{n+1} \end{vmatrix} \tag{2.9}$$

where

$$a_n = a_{n,n}, \qquad S_n(x) = \sum_{j=0}^{n} a_{j,n+1}\, g_j(x)$$

Bearing in mind that according to (2.7)

$$\int_a^b K_n(x, y)\, g_{n+1}(y)\, dy = \mu_{n+1}\, g_{n+1}(x)$$

and introducing designations

$$\Omega_n(x) = \sum_{j=0}^{n} a_{j,n+1} \int_a^b \Gamma_n(x, y)\, g_j(y)\, dy \tag{2.10}$$

$$T_n = \sum_{k=0}^{n}\sum_{m=0}^{n} a_{k,\,n+1}a_{m,\,n+1}\int_a^b\int_a^b \Gamma_n(x,y)\,g_k(x)\,g_m(y)\,dx\,dy \qquad (2.11)$$

we will have, in accordance with (2.4)

$$\Gamma_{n+1}(x,y) = \frac{\mu_{n+1}}{\Delta_n^*}\begin{vmatrix} \Gamma_n(x,y) & \mu_{n+1}^{-1}g_{n+1}(x) & \Omega_n(x) \\ \mu_{n+1}^{-1}g_{n+1}(y) & \mu_{n+1}^{-1}\lambda_{n+1} & -1 \\ \Omega_n(y) & -1 & a_{n+1}+T_n \end{vmatrix}. \qquad (2.12)$$

where

$$\Delta_n^* = \lambda_{n+1}(a_{n+1}+T_n) - \mu_{n+1}$$

After expanding the determinant, Formula (2.12) may be represented in the form

$$\Gamma_{n+1}(x,y) = \Gamma_n(x,y) - \frac{g_{n+1}(x)\,g_{n+1}(y)}{\mu_{n+1}\lambda_{n+1}} - \frac{1}{\Delta_n^*}[\Omega_n(x)\,g_{n+1}(y) +$$
$$+ g_{n+1}(x)\,\Omega_n(y) + \lambda_{n+1}\,\Omega_n(x)\,\Omega_n(y) + \lambda_{n+1}^{-1}g_{n+1}(x)\,g_{n+1}(y)] \qquad (2.13)$$

Setting

$$\Gamma_n(x,y) = \gamma(x,y) - \sum_{m=0}^{n}\frac{g_m(x)\,g_m(y)}{\mu_m\lambda_m} - \sum_{m=0}^{n}\sum_{k=0}^{n} A_{k,m}^{(n)}g_k(x)\,g_m(y) \qquad (2.14)$$

and utilizing Formulas (2.10) to (2.13), one finds that

$$\Gamma_{n+1}(x,y) = \gamma(x,y) - \sum_{m=0}^{n+1}\frac{g_m(x)g_m(y)}{\mu_m\lambda_m} - \sum_{m=0}^{n+1}\sum_{k=0}^{n+1} A_{k,m}^{(n+1)}g_k(x)\,g_m(y)$$

where the following formulas will hold true for the coefficents $A_{k,m}^{(n+1)}$

$$A_{m,k}^{(n+1)} = A_{m,k}^{(n)} - \frac{\lambda_{n+1}B_kB_m}{\Delta_n} \qquad (m,\,k\leqslant n)$$
$$A_{m,\,n+1}^{(n+1)} = A_{n+1,\,m}^{(n+1)} = \frac{B_m}{\Delta_n} \qquad (m\leqslant n), \qquad A_{n+1,\,n+1}^{(n+1)} = \frac{-1}{\Delta_n\lambda_{n+1}} \qquad (2.15)$$

where

$$B_m^{(n)} = \sum_{r=0}^{n} a_{r,\,n+1}\lambda_r A_{m,r}^{(n)}, \qquad \Delta_n = \mu_{n+1} - \lambda_{n+1}\Big(a_{n+1} - \sum_{k=0}^{n} a_{k,\,n+1}\lambda_k B_k^{(n)}\Big)$$

On the other hand, it can be readily shown that

$$\Gamma_0(x,y) = \gamma(x,y) - \frac{g_0(x)\,g_0(y)}{\mu_0\lambda_0} - \frac{g_0(x)\,g_0(y)}{\lambda_0(a_0\lambda_0-\mu_0)}$$

Hence

$$A_{0,0}^{(0)} = \lambda_0^{-1}(a_0\lambda_0-\mu_0)^{-1} \qquad (2.16)$$

Thus, knowing $A_{00}^{(0)}$ and using the recurrence formulas (2.15), we can construct the resolvent for the integral equation of the first kind with the kernel (2.8).

In the case of the integral equation of the second kind

$$\varphi(x) + \int_a^b k(x, y)\, \varphi(y)\, dy = f(x), \qquad \left[k(x, y) = \sum_{m=0}^{\infty} f_m^*(x)\, g_m^*(y)\right] \tag{2.17}$$

which can be written down in the form of an equation of the first kind

$$\int_a^b [\delta(x-y) + k(x, y)]\, \varphi(y)\, dy = f(x) \qquad [\delta(x) \quad \text{the impulse function}]$$

for the coefficients $A_{k,j}^{(n)}$, which define the resolvent

$$\Gamma_n(x, y) = \delta(x-y) - \sum_{k=0}^{n}\sum_{j=0}^{n} A_{k,j}^{(n)} f_k^*(x)\, g_j^*(y) \tag{2.18}$$

of the equation

$$\int_a^b \left[\delta(x-y) + \sum_{m=0}^{n} f_m^*(x)\, g_m^*(y)\right] \varphi_n(y)\, dy = f(x)$$

the recurrence formulas can be established in an analogous manner

$$A_{k,j}^{(n+1)} = A_{k,j}^{(n)} + \Delta_n^{-1} B_k^{(n)} C_j^{(n)} \qquad (k, j \leqslant n)$$
$$A_{n+1,j}^{(n+1)} = \frac{C_j^{(n)}}{\Delta_n}\ (j \leqslant n), \qquad A_{k,n+1}^{(n+1)} = \frac{B_k^{(n)}}{\Delta_n} \quad (k \leqslant n), \qquad A_{n+1,n+1}^{(n+1)} = \frac{1}{\Delta_n} \tag{2.19}$$

where

$$B_k^{(n)} = \sum_{j=0}^{n} A_{k,j}^{(n)} a_{n+1,j}, \qquad C_j^{(n)} = \sum_{k=0}^{n} A_{k,j}^{(n)} a_{k,n+1}, \qquad a_{k,j} = \int_a^b f_k^*(x)\, g_j^*(y)\, dy$$

$$\Delta_n = 1 + a_{n+1} + \sum_{k=0}^{n}\sum_{j=0}^{n} A_{k,j}^{(n)} a_{n+1,j} a_{k,n+1} \qquad (a_{k,k} = a_k)$$

Here, it is easily found that

$$A_{0,0}^{(0)} = (1 + a_0)^{-1}$$

3. Let us indicate some possible ways of applying the above results to the construction of approximate solutions of certain integral equations of mathematical physics.

Suppose it is required to construct an approximate solution of the integral equation

$$\varphi(x) + \frac{\lambda}{\pi}\int_0^a K_0(|x-s|)\, \varphi(s)\, ds = f(x) \qquad (0 \leqslant x \leqslant a) \tag{3.1}$$

Retaining the $(n+1)$th term in Expansion (1.14), we obtain the following approximate representation for the kernel of the integral equation

$$K_0(|x-s|) \approx \pi e^{-x-s} \sum_{m=0}^{n} L_m^{-1/2}(2x)\, L_m^{-1/2}(2s) \qquad (x, s \geqslant 0)$$

by means of which we reduce (3.1) to the integral equation of (2.17) type. Therefore, if the solution of Equation (3.1) with the kernel defined by the above formula is designated by $\varphi_n(x)$ ($(n+1)$th approximation of the exact solution), then according to (2.18) we will have

$$\varphi_n(x) = f(x) - \int_0^a \Gamma_n^*(x, y)\, f(y)\, dy \tag{3.2}$$

where

$$\Gamma_n^*(x,y)=\sum_{k=0}^{n}\sum_{j=0}^{n}A_{k,j}^{(n)}L_k^{-1/2}(2x)\,L_j^{-1/2}(2y)\,e^{-x-y}$$

For the coefficients $A_{k,j}^{(n)}$ the recurrence formulas (2.2) are valid, in which by virtue of

$$f_k^*(x)=g_k^*(x)=\sqrt{\lambda}e^{-x}L_k^{-1/2}(2x)$$

one should set

$$C_j^{(n)}=B_j^{(n)},\qquad A_{k,j}^{(n)}=A_{j,k}^{(n)},\qquad a_{k,j}=\lambda\int_0^a e^{-2x}L_k^{-1/2}(2x)\,L_j^{-1/2}(2x)\,dx \tag{3.3}$$

The last integral can be presented in the form

$$a_{k,j}=\lambda\,[a_{k,j}^{\infty}-e^{-2a}P_{kj}(2a)] \tag{3.4}$$

The numbers $a_{k,j}^{\infty}$ and the polynomial $P_{kj}(b)$ are defined by Formulas

$$a_{k,j}^{\infty}=\frac{1}{2}\int_0^{\infty}e^{-t}L_k^{-1/2}(t)\,L_j^{-1/2}(t)\,dt=$$

$$=\frac{(-1)^k(2j-1)!!}{2^{j+k+1}k!}\sum_{r=0}^{j}\frac{(-2)^r(1+2r)}{(j-r)!\,[2(r-k)+1]!!} \tag{3.5}$$

$$P_{kj}(b)=\frac{e^b}{2}\int_b^{\infty}e^{-t}L_k^{-1/2}(t)\,L_j^{-1/2}(t)\,dt=\sum_{m=0}^{k+j}c_mL_m^{-1/2}(b) \tag{3.6}$$

where

$$2c_m=\frac{(2k-1)!!\,(2j-1)!!}{(2m+1)!!\,2^{k+j-m}}\sum_{r=0}^{k}\sum_{s=\max(0,\,m-r)}^{j}\frac{(-1)^{r+s+m}(2r+2s+1)!}{(2r)!\,(2s)!\,(r+s-m)!\,(k-r)!\,(j-s)!}$$

In order to verify the second equality in Formula (3.5) one should replace $L_k^{-1/2}(t)$ by its expression in terms of weight function and intergate by parts k times. For the proof of the second equality in (3.6) $P_{kj}(b)$ should be expanded into a series of Laguerre's polynomials.

Formula (3.4) and, therefore, the formula for the approximate solution of the integral equation (3.2) are simplified considerably for $a=\infty$, since in that case $a_{k,j}=\lambda a_{k,j}^{\infty}$.

Grinberg and Fok [7] have shown that the problem of coastal refraction of electromagnetic waves can be reduced to the integral equation (3.1) for $a=\infty$. The authors have obtained the exact solution of the above integral equation in the form of a rather complex double quadrature; consequently, they proposed an approximate solution valid only if the wave incidence angle is not too oblique. The approximate solution proposed in this paper does not contain any quadratures and may be used, generally speaking, for any parameters of Equation (3.1).

4. Let us apply the results obtained in the first two sections to the construction of an approximate solution of the contact problem of a semi-infinite plate with the elastic half-space. It has been shown in [8] that if a semi-infinite plate with the flexural rigidity D is acted upon by a unit force at point $y=0$, $x=b$, then the Fourier transform $p_\lambda(x)$ of the contact stress [9] must satisfy the integral equation

$$\int_0^\infty \left\{\frac{1}{\pi} K_0(\lambda \mid x - \xi \mid) + \frac{c^3}{\lambda^3} G[\lambda(x-\xi)]\right\} p_\lambda(\xi)\, d\xi = f_\lambda(x) \qquad (x, \lambda \geqslant 0) \quad (4.1)$$

where

$$f_\lambda(x) = c^3 (A_0 + A_1 \lambda x) e^{-\lambda x} + \frac{c^3}{\lambda^3} G[\lambda(x-b)] \qquad (4.2)$$

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-its}}{(1+s^2)^2} ds, \qquad c^3 = \frac{E_1}{2(1-\nu_1^2) D} \qquad (4.3)$$

Here, A_0 and A_1 are arbitrary constants, ν_1, E_1 are the Poisson's ratio and modulus of elasticity of the half space.

After $p_\lambda(x)$ is found, the Fourier transform $w_\lambda(x)$ of the plate deflections may be determined from one of the Formulas [8 and 9]

$$w_\lambda(x) = \frac{\theta_1}{\pi} \int_0^\infty K_0(\lambda \mid x - \xi \mid) p_\lambda(\xi)\, d\xi \qquad \left(\theta_1 = \frac{2(1-\nu_1^2)}{E_1}\right) \qquad (4.4)$$

$$D w_\lambda(x) = (A_0 + A_1 \lambda x) e^{-\lambda x} + \frac{1}{\lambda^3}\left[G[\lambda(x-b)] - \int_0^\infty G[\lambda(x-\xi)] p_\lambda(\xi)\, d\xi\right] \quad (x \geqslant 0) \qquad (4.5)$$

The arbitrary constants A_0 and A_1 should be found from the free edge conditions for the plate [8 and 9]

$$w_\lambda^{(2)}(+0) - \lambda^2 \nu w_\lambda(+0) = 0, \quad w_\lambda^{(3)}(+0) - \lambda^2 (2-\nu) w_\lambda^{(1)}(+0) = 0 \quad (4.6)$$

Multiplying both sides of Equation (4.1) by $\sqrt{2/\pi}$ and making a substitution

$$\lambda x = t, \quad \lambda \xi = \tau, \quad \frac{1}{\lambda} p_\lambda\left(\frac{\tau}{\lambda}\right) = p(\tau), \quad \left(\frac{2}{\pi}\right)^{1/2} f_\lambda\left(\frac{t}{\lambda}\right) = f^*(t) \quad (4.7)$$

we obtain

$$\int_0^\infty \left[\frac{\sqrt{2}}{\pi^{3/2}} K_0(\mid t - \tau \mid) + \left(\frac{2}{\pi}\right)^{1/2} \frac{c^3}{\lambda^3} G(t-\tau)\right] p(\tau)\, d\tau = f^*(t) \qquad (4.8)$$

The resulting expansions are

$$G(t-\tau) = e^{-(t+\tau)} \sum_{k=0}^\infty \sum_{m=0}^\infty b_{km} L_k^{-1/2}(2t) L_m^{-1/2}(2\tau). \qquad (4.9)$$

In view of the orthogonality of the Laguerre's polynomials, we can write down that

$$\frac{b_{km}}{2} = \frac{k!\, m!}{\Gamma(1/2 + k)\, \Gamma(1/2 + m)} \int_0^\infty L_k^{-1/2}(2t) I_m(t) e^{-t} \frac{dt}{\sqrt{t}} \qquad (4.10)$$

where

$$I_m(t) = \int_0^\infty G(t-\tau) L_m^{-1/2}(2\tau) e^{-\tau} \frac{d\tau}{\sqrt{\tau}} \qquad (4.11)$$

By virtue of the convolution theorem for Fourier transforms, and taking into account (4.3), the last integral may be put into the form

$$I_m(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi_m(u)\, e^{-iut}\, du}{(1+u^2)^2} \tag{4.12}$$

where

$$\Phi_m(u) = \int_0^{\infty} L_m^{-1/2}(2\tau)\, e^{-\tau(1-iu)} \frac{d\tau}{\sqrt{\tau}} = \frac{(-1)^m\, \Gamma(m+1/2)}{m!} \frac{(1+iu)^m}{(1-iu)^{m+1/2}} \tag{4.13}$$

In evaluating the integral contained in (4.13) Formula 7.414 (8) from [3] was used. Substituting (4.12) into (4.10), after the interchange of integrals and application of Formula (4.13), we obtain

$$b_{km} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(-1)^{k+m}\, du}{(1+iu)^{k-m+5/2}(1-iu)^{m-k+5/2}} = \frac{12}{\pi} \frac{1}{9-4(k-m)^2} \frac{1}{1-4(k-m)^2} \tag{4.14}$$

The last equality was established with the aid of Formula 8381 (1) from [3], and the well known properties of Euler's gamma-function.

If both sides of Equation (4.8) are multiplied by $t^{-1/4}$, and if we set

$$k(t,\tau) = \frac{1}{\pi}\left(\frac{2}{\pi}\right)^{1/2} (t\tau)^{-1/4} K_0(|t-\tau|), \qquad g_m(t) = \left(\frac{2}{\pi t}\right)^{1/4} e^{-t} L_m^{-1/2}(2t) \tag{4.15}$$

$$t^{1/4}\,[p(t), f(t)] = \varphi(t), f^*(t); \qquad a_{k,m} = -\sigma^3 \alpha_{k-m}, \quad \sigma^3 = \frac{c^3}{\lambda^3}\frac{4}{\pi}$$

$$\alpha_n = \frac{4}{(1-4n^2)(9-4n^2)} \tag{4.16}$$

then, in view of (4.9), we obtain

$$\int_0^{\infty} \left[k(t,\tau) - \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} a_{k,m}\, g_k(t)\, g_m(\tau)\right] \varphi(\tau)\, d\tau = f(t) \tag{4.17}$$

where, due to (1.13) and (4.15), the relations (2.7) will be valid (for $a = 0$, $b = \infty$), and the following relation as well

$$\mu_m = \lambda_m = (2m!!)^{-1}(2m-1)!! \qquad (\mu_0 = \lambda_0 = 1) \tag{4.18}$$

Using Formulas (4.2), (4.7), (4.9), (4.14) and (4.16) the right-hand side of the integral equation (4.17) can be written down in the form

$$f(t) = A_0^* g_0(t) - A_1^* g_1(t) + \sigma^3 \sum_{k=0}^{\infty} g_k(t) \sum_{m=0}^{\infty} \alpha_{k-m}\, g_m(\beta)\, \beta^{1/4} \tag{4.19}$$

where

$$\beta = \lambda b, \qquad A_0^* = (2/\pi)^{1/4} c^3 (A_0 + 1/4 A_1), \qquad A_1^* = 1/2 c^3 (2/\pi)^{1/4} A_1 \tag{4.20}$$

According to the theory given in Section 2, the solution of the integral equation (4.17) for the case when $(n+1)$th term of the infinite sum contained in it is retained, will be determined by Formula

$$\varphi_n(t) = \int_0^{\infty} \Gamma_n(t,\tau)\, f(\tau)\, d\tau$$

or, after (2.14) is substituted in the above and in view of (4.19)

$$\varphi_n(t) = -A_0^* \sum_{k=0}^{n} A_{k,0}^{(n)} g_k(t) + \frac{1}{2} A_1^* \sum_{k=0}^{n} A_{k,1}^{(n)} g_k(t) - \sigma^3 \sum_{k=0}^{n} \omega_k^{(n)}(\beta)\, g_k(t) \tag{4.21}$$

where

$$\omega_k^{(n)}(\beta) = \sum_{m=0}^{n} \sum_{r=0}^{n} \lambda_m \alpha_{m-r} A_{k,m}^{(n)} g_r(\beta) \sqrt[4]{\beta} \tag{4.22}$$

Therefore, due to (4.15), (4.16) and (4.7), the Fourier transform $p_\lambda^n(x)$ of the contact stress in the $(n+1)$th approximation is determined by Formula

$$p_\lambda^n(x) = -\left(\frac{2}{\pi}\right)^{1/4}\left(\frac{\lambda}{x}\right)^{1/2} e^{-\lambda x}\Big\{A_0^* \sum_{k=0}^{n} A_{k,0}^{(n)} L_k^{-1/2}(2\lambda x) -$$

$$-\frac{1}{2} A_1^* \sum_{k=0}^{n} A_{k,1}^{(n)} L_k^{-1/2}(2\lambda x) + \sigma^3 \sum_{k=0}^{n} \omega_k^{(n)}(\lambda b) L_k^{-1/2}(2\lambda x)\Big\} \qquad (x \geqslant 0) \tag{4.23}$$

It remains now to find the values of the arbitrary constants A_0^* and A_1^* by satisfying the free edge conditions (4.6).

The substitution of (4.23) into (4.4) with the use of relation (1.13) gives

$$w_\lambda^n(x) = -\left(\frac{2}{\pi}\right)^{1/4} \theta_1 e^{-\lambda x}\Big[A_0^* \sum_{k=0}^{n} A_{k,0}^{(n)} \lambda_k L_k^{-1/2}(2\lambda x) -$$

$$-\frac{1}{2} A_1^* \sum_{k=0}^{n} A_{k,1}^{(n)} \lambda_k L_k^{-1/2}(2\lambda x) + \sigma^3 \sum_{k=0}^{n} \omega_k^{(n)}(\lambda b) \lambda_k L_k^{-1/2}(2\lambda x)\Big] \qquad (x \geqslant 0) \tag{4.24}$$

With the aid of formula 8.971 (2) from [3] it is easy to show that

$$\frac{d^n}{dx^n} e^{-\lambda x} L_k^{-1/2}(2\lambda x) = (-\lambda)^n e^{-\lambda x} \sum_{j=0}^{\min(n,\,k)} \binom{n}{j} 2^j L_{k-j}^{-1/2+j}(2\lambda x)$$

and, hence,

$$\left[\frac{d^n}{dx^n} e^{-\lambda x} L_k^{-1/2}(2\lambda x)\right]_{x=0} = (-\lambda)^n c_k^{(n)} \left[c_k^{(n)} = \sum_{j=0}^{\min(n,\,k)} 2^j \binom{n}{j}\binom{k-1/2}{k-j}\right] \tag{4.25}$$

Taking into account (4.24) and (4.25), we find

$$\left[\frac{d^r}{dx^r} w_\lambda^n(x)\right]_{x=0} = -(-\lambda)^r \theta_1 \left(\frac{\pi}{2}\right)^{1/4}\Big\{A_0^* \sum_{k=0}^{n} A_{k,0}^{(n)} \lambda_k c_k^{(r)} -$$

$$-\frac{1}{2} A_1^* \sum_{k=0}^{n} A_{k,1}^{(n)} \lambda_k c_k^{(r)} + \sigma^3 \sum_{k=0}^{n} \omega_k^{(n)}(\lambda b) \lambda_k c_k^{(r)}\Big\} \tag{4.26}$$

Substituting the obtained values of the derivatives of $w_\lambda^n(x)$ for $x = 0$ into the free edge conditions (4.6), we find

$$A_0^* = \sigma^3 \frac{B_{1,0}^{(n)} \Omega_1^{(n)}(\lambda b) - B_{1,1}^{(n)} \Omega_0^{(n)}(\lambda b)}{B_{0,0}^{(n)} B_{1,1}^{(n)} - B_{0,1}^{(n)} B_{1,0}^{(n)}}, \quad -\frac{A_1^*}{2} = \sigma^3 \frac{B_{0,1}^{(n)} \Omega_0^{(n)}(\lambda b) - B_{0,0}^{(n)} \Omega_1^{(n)}(\lambda b)}{B^{(n)} B_{1,1}^{(n)} - B_{0,1}^{(n)} B_{1,0}^{(n)}}$$

where

$$B_{j,r}^{(n)} = \sum_{k=0}^{n} A_{k,j}^{(n)} \lambda_k d_k^{(r)}, \qquad \Omega_r^{(n)}(\beta) = \sum_{k=0}^{n} \omega_k^{(n)}(\beta)\lambda_k d_k^{(r)}$$

$$d_k^{(0)} = c_k^{(2)} - \nu c_k^{(0)}, \qquad d_k^{(1)} = c_k^{(3)} - (2-\nu)c_k^{(1)}$$

A substitution of (4.27) into (4.23) and (4.24) will yield $p_\lambda^n(x)$, $w_\lambda^n(x)$, after which, by means of Formulas (from [8 and 9])

$$p_n(x, y), w_n(x, y) = \frac{1}{\pi}\int_0^\infty [p_\lambda^n(x), \qquad w_\lambda^n(x)] \cos \lambda y \, d\lambda \tag{4.28}$$

we obtain, in the $(n+1)$th approximation, the contact stress $p_n(x, y)$ and the deflections of the semi-infinite plate $w_n(x, y)$, acted upon by a unit concentrated force at point $y = 0$, $x = b \geqslant 0$

The quantities $A_{k,m}^{(j)}$, contained in Formulas (4.23) and (4.24) should be evaluated by means of recurrence formulas (2.15), taking into account (4.16) and (4.18), and for the first approximation ($n = 0$), in accordance with (2.16) we have

$$A_{0,0}^{(0)} = (1 + {}^1/_3\sigma^3)^{-1}$$

Thus, the application of the proposed approximate method to the contact problem of a semi-infinite plate yields, for any degree of approximation, the final formulas (4.26), (4.23), (4.24) and (4.27), containing only one quadrature instead of the fourfold quadrature given by the exact method [8 and 9].

In conclusion, we note that using the results of the first two sections, and Grinberg's results [10], one can work out analogous approximate methods of solution of integral equations, to which the diffraction problem of electromagnetic waves on a strip of finite width, and the three-dimensional problem of bending of an infinite beam on the elastic half-space are reduced.

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